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EVOLUTION OF THE ROTATIONAL MOTION OF A VISCOELASTIC SPHERE IN A CENTRAL NEWTONIAN FORCE FIELD*

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The motion of a viscoelastic sphere (a planet), whose centre of mass moves along a circular orbit in a central Newtonian force field, is considered. Using the method of separation of motions and averaging, approximate equations are obtained which define the rotary motion of the sphere in canonical Andoyer variables, and the evolution of such motion is investigated.

Approximate equations were obtained earlier /1/ which described the translational-rotational motion of a viscoelastic sphere in a central Newtonian force field, steady motions were determined and their stability investigated. Models of tidal phenomena which induce evolution of the rotational motion of planets were studied in /2—4/. The equation of the change of angular momentum of a viscoelastic sphere obtained in /1/ corresponds to the model of tidal effects, when the "tidal hump" is turned relative to the line attracting the centre of the planet mass, by an angle proportional to the angular velocity of planet rotation in orbital axes /4/. In addition, this equation contains the moment due to the deformation of the planet by the action of centrifugal inertia forces producing the planet regular precession.

Let a homogeneous isotropic elastic sphere occupy a region Ω in the inertial system of coordinates $O\,\xi_1\xi_2\xi_3$ in the naturally undeformed state, and let the motion of the sphere be defined by the one-parameter group

$$g^t\colon \Omega \to E^3,\ \xi = \xi(\mathbf{r}, t),\ \mathbf{r} \in \Omega,\ t \in R^1$$

Following /1/, we represent the vector field $\xi(\mathbf{r}, t)$ in the form (here and subsequently, unless otherwise stated, the integrals are taken over the region Ω)

$$\xi(\mathbf{r}, t) = \mathbf{R}(t) + O(t)\,(\mathbf{r} + \mathbf{u}(\mathbf{r}, t)) \tag{1}$$
$$\mathbf{R} = \frac{1}{M}\int \xi\rho\,dx, \quad \int \mathbf{u}\rho\,dx = \int \operatorname{rot}\mathbf{u}\,dx = 0$$
$$M = \int \rho dx,\ \rho = \text{const}$$

where ρ is the density of the sphere. Conditions (1) uniquely define the radius vector $\mathbf{R}(t)$ of the centre of mass of the sphere, the system of coordinates $Cx_1x_2x_3$ relative to which the sphere does not rotate in the integral sense. The operator $O(t)$ belongs to the group of rotations of three-dimensional space and determines the transition from the system of coordinates $Cx_1x_2x_3$ to Koenig's system $C\,\xi_1\xi_2\xi_3$. We assume that the quantities $\partial u_i/\partial x_j\,(i, j = 1, 2, 3)$ are small ($|\partial u_i/\partial x_j| \ll 1$) and that the deformed state of the sphere is defined by the classical theory of elasticity of small deformations; in particular, the functional of potential energy of elastic deformation has the form /2, 5/

$$E[\mathbf{u}] = \int a'\,(\Sigma_1^2 - a_1'\Sigma_2^2)\,dx,\ a' > 0,\ 0 < a_1' < 3 \tag{2}$$
$$a' = \frac{E(1-\nu)}{2(1+\nu)(1-2\nu)}, \quad a_1' = \frac{2(1-2\nu)}{1-\nu}$$
$$\Sigma_1 = \sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}, \quad \Sigma_2 = \sum_{i<j}^{3}\left[\left(\frac{\partial u_i}{\partial x_i}\right)\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{1}{4}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)^2\right]$$

where E, ν are Young's modulus of elasticity and Poisson's ratio, respectively.

The potential of gravitational forces and forces of inertia of translational motion (the translation motion of the system of coordinates $C\,\xi_1\xi_2\xi_3$) is given by the functional

$$\Pi_1 = -\int \mu\rho\,\{[(\mathbf{R} + O(\mathbf{r} + \mathbf{u}))^2]^{-1/2} + R^{-3}\mathbf{R}O\mathbf{u}\}\,dx \tag{3}$$

where μ is the gravitational constant. The gravitational intersection of particles of the body is defined by the potential-energy functional

$$\Pi_2 = \int g\,\frac{\mathbf{r}}{r_0}\,\mathbf{u}\,dx$$

(g is the acceleration due to gravity on the sphere surface and r_0 is the sphere radius) which produces spherically symmetric deformation of the sphere /5/. It will be shown

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subsequently that these deformations lead to the evolution of the sphere rotation.

Consider the formulation of the problem when the centre of mass of the sphere describes a Keppler circular orbit of radius R about the attracting centre O. Then

$$\mathbf{R} = R\mathbf{R}^0,\ \mathbf{R}^0 = \cos\vartheta\,\xi_1 + \sin\vartheta\,\xi_2,\ \vartheta = \sqrt{\mu/R^3}\,t$$

The configurational manifold of the system is then

$$W = SO(3) \times V_0$$
$$V_0 = \{\mathbf{u} : \mathbf{u} \in (W_2^1(\Omega))^3,\ \int \mathbf{u}\,dx = \int \operatorname{rot}\mathbf{u}\,dx = 0\}$$

The functional of the kinetic energy in the motion relative to the system of coordinates $C\xi_1\xi_2\xi_3$ is

$$T = \frac{1}{2}\int [\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}) + \mathbf{u}^{\cdot}]^2 \rho\,dx$$

We introduce on the tangential stratification of the group of rotations SO (3) the Andoyer canonical variables $I_i,\ \varphi_i\ (i = 1, 2, 3)$ /6/ and obtain the Routh functional of the form

$$R_*[\mathbf{I}, \boldsymbol{\varphi}, \mathbf{u}^{\cdot}, \mathbf{u}] = \frac{1}{2}(\mathbf{G} - \mathbf{G}_u, J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u)) - \tag{4}$$
$$\frac{1}{2}\int \mathbf{u}^{\cdot 2}\rho\,dx + E[\mathbf{u}] + \Pi_1 + \Pi_2$$
$$\mathbf{G} = \nabla_\omega T = \int (\mathbf{r} + \mathbf{u}) \times [\boldsymbol{\omega} \times (\mathbf{r} + \mathbf{u}) + \mathbf{u}^{\cdot}]\rho\,dx$$
$$\mathbf{G}_u = \int (\mathbf{r} + \mathbf{u}) \times \mathbf{u}^{\cdot}\rho\,dx$$

where $J^{-1}[\mathbf{u}]$ is the inertia operator in the system of coordinates $Cx_1x_2x_3$, and the matrix $O(t)$ in Π_1, and the vector of angular momentum G are expressed in terms of Andoyer variables /7/

$$\mathbf{G} = (\sqrt{I_2^2 - I_1^2}\sin\varphi_1,\ \sqrt{I_2^2 - I_1^2}\cos\varphi_1,\ I_1) \tag{5}$$
$$O(t) = \Gamma_3(\varphi_3)\,\Gamma_1(\delta_1)\,\Gamma_3(\varphi_2)\,\Gamma_1(\delta_2)\,\Gamma_3(\varphi_1)$$
$$\Gamma_3(\alpha) = \begin{Vmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{Vmatrix},\quad \Gamma_1(\alpha) = \begin{Vmatrix}1 & 0 & \\ 0 & \cos\alpha & -\sin\alpha\\ 0 & \sin\alpha & \cos\alpha\end{Vmatrix}$$

This problem contains a "large" parameter, the characteristic of elastic sphere rigidity (Young's modulus is assumed large and, consequently, the sphere deformations are small). In the limit, for infinite rigidity, the sphere deformation will be zero $(\cdot \equiv 0)$ and the Routh functional of the unperturbed problem will take the form

$$R_0 = I_2^2/(2A)$$

where A is the moment of inertia of the undeformed sphere relative to its diameter.

The unperturbed motion of the sphere is a uniform rotation around one of its diameters with angular velocity $\varphi_2^{\cdot} = I_2A^{-1}$. When the rigidity is finite the Routh equations take the form

$$I_i^{\cdot} = -\nabla_{\varphi_i}R_*,\ \varphi_i^{\cdot} = \nabla_{I_i}R_*\ (i = 1, 2, 3) \tag{6}$$
$$\int \left[\left(\frac{d}{dt}\nabla_{u^{\cdot}}R_* - \nabla_u R_* - \nabla_{u^{\cdot}}D + \lambda_1\right)\delta\mathbf{u} + \lambda_2\operatorname{rot}\delta\mathbf{u}\right]dx = 0$$
$$\forall\delta\mathbf{u} \in V,\ V = \{\mathbf{u} : \mathbf{u} \in (W_2^1(\Omega))^3\}$$

The second equation of (6) is written in the form of the d'Alembert-Lagrange variational principle and contains two undetermined multipliers λ_1 and λ_2. The gradient of the dissipative functional $\nabla_{u^{\cdot}}D$ defines viscous dissipative forces. We will assume that the dissipative functional $D[\mathbf{u}^{\cdot}]$ is proportional to that of the potential energy of elastic deformations, if in the latter the components of the small deformation tensor is replaced by respective components of the velocity deformation tensor, i.e. $D[\mathbf{u}^{\cdot}] = \chi E[\mathbf{u}^{\cdot}]$ [1].

The second equation of (6), taking (4) into account has the form

$$\int \Big\{\frac{d}{dt}[(\mathbf{r} + \mathbf{u}) \times J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u)]\rho - \rho\mathbf{u}^{\cdot\cdot} - \tag{7}$$
$$\frac{1}{2}(\mathbf{G} - \mathbf{G}_u, \nabla_u J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u)) - [\mathbf{u}^{\cdot} \times J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u)]\rho -$$
$$\nabla E[\mathbf{u}] - \chi\nabla_{u^{\cdot}}E[\mathbf{u}^{\cdot}] + \mu\rho[(\mathbf{R} + O(\mathbf{r} + \mathbf{u}))^2]^{-3/2}O^{-1}(\mathbf{R} +$$
$$O(\mathbf{r} + \mathbf{u})) - \mu\rho R^{-3}O^{-1}\mathbf{R} + gr_0^{-1}\mathbf{r} + \lambda_1\Big\}\delta\mathbf{u}\,dx +$$
$$\int_{\delta\Omega}(\lambda_2 \times \mathbf{n})\,\delta\mathbf{u}\,d\sigma = 0$$
$$\forall\delta\mathbf{u} \in (W_2^1(\Omega))^3$$

The last integral in (7) has been transformed by Gauss's formula and $\mathbf{n}$ is the normal to the sphere surface. The solution of Eq.(7) is sought in the form of series in the small parameter $\varepsilon = E^{-1}$

$$\mathbf{u} = \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \ldots \tag{8}$$

Since subsequently we shall use the method of separation of motions and of averaging /8/, it is sufficient to determine the function $\mathbf{u}_1(\mathbf{r}, t)$ which satisfies the equation

$$\int \Big\{ \rho \mathbf{r} \times J^{-1}[0]\, \mathbf{G}^{\cdot} - \frac{1}{2}(\mathbf{G}, J^{-1}[0]\, \nabla_u J_1[\mathbf{u}]\, J^{-1}[0]\, \mathbf{G}) - \varepsilon \nabla E[\mathbf{u}_1] - \varepsilon \chi \nabla E[\mathbf{u}_1^{\cdot}] + \mu\rho R^{-3}\mathbf{r} - 3\mu\rho R^{-3}(\mathbf{R}^\circ, O\mathbf{r})\, O^{-1}\mathbf{R}^\circ + \boldsymbol{\lambda}_1 \Big\} \delta\mathbf{u}\, dx + \int_{\partial\Omega} (\boldsymbol{\lambda}_2 \times \mathbf{n})\, \delta\mathbf{u}\, d\sigma = 0 \tag{9}$$

In deriving Eq.(9) we used the equation

$$J^{-1}[\mathbf{u}] = (J[0] + J_1[\mathbf{u}] + J_2[\mathbf{u}])^{-1} = J^{-1}[0] - J^{-1}[0]\, J_1[\mathbf{u}]\, J^{-1}[0] + \ldots$$

where

$$J^{-1}[0] = A^{-1}\ \mathrm{diag}\ \{1,1,1\}, \quad J_1[\mathbf{u}] = dJ[\lambda\mathbf{u}]/d\lambda\,|_{\lambda=0}$$

is linear with respect to the component $\mathbf{u}$ of the inertia operator of the deformed sphere. Note that $\nabla_u J_1[\mathbf{u}]$ is independent of $\mathbf{u}$.

The following equations are also valid:

$$(\mathbf{G}, J_1[\mathbf{u}]\,\mathbf{G}) = 2\,[\mathbf{r} \times \mathbf{G}]\,[\mathbf{u} \times \mathbf{G}]\,\rho dx$$
$$(\mathbf{G}, \nabla_u J_1[\mathbf{u}]\,\mathbf{G}) = -2\rho \mathbf{G} \times [\mathbf{G} \times \mathbf{r}]$$

If in the variational equation (9) we set $\delta\mathbf{u} = \mathbf{a} \in E^3$ or $\delta\mathbf{u} = \delta\boldsymbol{\alpha} \times (\mathbf{r} + \mathbf{u})$ and $\delta\boldsymbol{\alpha} \in E^3$ (possible displacements correspond to the group of rotation-displacements of the three-dimensional space), we find that $\lambda_1 = \lambda_2 = 0$ /1/. Note that in the unperturbed motion $\mathbf{G}^{\cdot} = 0$, and we rewrite (9) in the form /1, 5/

$$-\varepsilon \nabla E[\mathbf{u}_1 + \chi \mathbf{u}_1^{\cdot}] = A^{-2}\rho \mathbf{G} \times [\mathbf{G} \times \mathbf{r}] + a\mathbf{r} - 3\mu\rho R^{-3} O^{-1}\mathbf{R}^\circ \times [O^{-1}\mathbf{R}^\circ \times \mathbf{r}], \quad a = -2\mu\rho R^{-3} + g r_0^{-1} \tag{10}$$
$$-\varepsilon \nabla E[\mathbf{u}_1] = \Big[\Delta \mathbf{u}_1 + \frac{1}{1-2\nu}\,\mathrm{grad\,div}\,\mathbf{u}_1\Big] \frac{1}{2(1+\nu)}$$

Equation (10) defines the quasisteady process of deformation of a viscoelastic sphere. The boundary conditions for the function $\mathbf{u}_1$ are formulated on $\partial\Omega$ in the form $\sigma \cdot \mathbf{n} = 0$ where σ is the stress tensor. Since (10) is linear and its first two terms on the right side are independent of time, its solution can be represented in the form of the sum of three functions $\mathbf{u}_1 = \mathbf{u}_{11} + \mathbf{u}_{12} + \mathbf{u}_{13}$ which satisfy the equations

$$-\varepsilon \nabla E[\mathbf{u}_{11}] = A^{-2}\rho \mathbf{G} \times [\mathbf{G} \times \mathbf{r}], \quad -\varepsilon \nabla E[\mathbf{u}_{12}] = a\mathbf{r} \tag{11}$$
$$-\varepsilon \nabla E[\mathbf{u}_{13} + \chi \mathbf{u}_{13}^{\cdot}] = -3\mu\rho R^{-3}[O^{-1}\mathbf{R}^0 \times [O^{-1}\mathbf{R}^0 \times \mathbf{r}]]$$

The solutions of (11) have the form /1, 2/

$$\mathbf{u}_{11}(r) = \rho I_2^2 A^{-2} \Gamma_3(-\varphi_1)\ \Gamma_1(-\delta_2)\, \mathbf{u}^*(\Gamma_1(\delta_2)\Gamma_3(\varphi_1)\,\mathbf{r}) \tag{12}$$
$$\mathbf{u}_{12}(r) = \frac{(1+\nu)(1-2\nu)}{10(1-\nu)}\Big[\mathbf{r}^2 - \frac{3-\nu}{1+\nu} r_0^2\Big] a\mathbf{r}$$
$$\mathbf{u}_{13}(\mathbf{r}, t) = \sum_{n=0}^{\infty} (-\chi)^n \frac{\partial^n \mathbf{u}_{130}(\mathbf{r}, t)}{\partial t^n}$$
$$\mathbf{u}_{130}(\mathbf{r}, t) = -3\mu\rho R^{-3} O_1^{-1}(t)\, \mathbf{u}^*(O_1(t)\,\mathbf{r})$$
$$O_1(t) = \Gamma_0(\vartheta)\, O(t), \quad \Gamma_0(\vartheta) = \begin{Vmatrix} -\sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 1 \\ \cos\vartheta & \sin\vartheta & 0 \end{Vmatrix}$$
$$\mathbf{u}^*(\mathbf{r}) = [(B_1\mathbf{r}, \mathbf{r})\, B_2\mathbf{r} + (B_3\mathbf{r}, \mathbf{r})\, B_4\mathbf{r} + B_5]\,\mathbf{r}$$
$$B_1 = \mathrm{diag}\,\{b_1, b_1, b_2\},\ B_2 = \mathrm{diag}\,\{1, 1, 0\},\ B_3 = \mathrm{diag}\,\{a_1, a_1, a_2\}$$
$$B_4 = \mathrm{diag}\,\{0, 0, 1\},\ B_5 = \mathrm{diag}\,\{c_1, c_1, c_2\}$$
$$b_1 = -(4 - 3\nu - 5\nu^2)\,\psi(\nu), \quad b_2 = -(9 - 8\nu - 5\nu^2)\,\psi(\nu)$$
$$a_1 = 2(3-\nu)\,\psi(\nu),\ a_2 = (1+3\nu)\,\psi(\nu)$$
$$\psi(\nu) = \frac{1+\nu}{5(1-\nu)(5\nu+7)}$$
$$c_1 = r_0^2 \frac{12 - 8\nu - 12\nu^2}{35 - 10\nu - 25\nu^2}, \quad c_2 = -r_0^2 \frac{3 + 18\nu - 3\nu^2 - 10\nu^3}{35 - 10\nu - 25\nu^2}$$

The function $\mathbf{u}_{11}(\mathbf{r})$ defines the axisymmetric elastic deformation of the sphere (compression of the sphere along the axis of rotation) by the action of centrifugal forces of inertia generated by the sphere's own rotation. The functions $\mathbf{u}_{12}(\mathbf{r})$ correspond to spherically symmetric deformation of the sphere, produced by the inner gravitational and the external gravitational field.

The external gravitational field also determines the unsteady deformation of the sphere (gravitational tides) which is defined by the function $\mathbf{u}_{13}(\mathbf{r}, t)$. In an orbital system of coordinates $Cxyz$ (the Cz axis coincides with the direction to the attracting centre, the Cx axis is tangent to the orbit, and the Cy axis is orthogonal to the orbital plane), the function $\mathbf{u}_{13}(\mathbf{r}, t)$ is represented by the first two terms of series (12) and has the form /2/

$$\mathbf{u}_{13}'(\mathbf{r}', t) = -3\mu\rho R^{-3}\{\mathbf{u}^*(\mathbf{r}') + \chi[(B_1\mathbf{r}', \mathbf{r}')(SB_2 - B_2S) + (B_3, \mathbf{r}', \mathbf{r}')(SB_4 - B_4S) + r_0^2(SB_5 - B_5S)]\mathbf{r}' - 2\chi[(B_1 S\mathbf{r}', \mathbf{r}')B_2 + (B_3S\mathbf{r}', \mathbf{r}')B_4]\mathbf{r}', \quad S = O_1^{\cdot}O_1^{-1} \tag{13}$$

where S is a skew symmetric matrix that defines the angular velocity of the sphere $\mathbf{\Omega}^*$ relative to the orbital system of coordinates $Cxyz$. The series in (12) that defines $\mathbf{u}_{13}(\mathbf{r}, t)$ converges, if $\chi\,|\mathbf{\Omega}^*| < 1$, while expression (13) approximates $\mathbf{u}_{13}(\mathbf{r}, t)$ quite well provided $\chi\,|\mathbf{\Omega}^*| \ll 1$, which is henceforth assumed.

We substitute the function $\mathbf{u}(\mathbf{r}, t) \approx \varepsilon\mathbf{u}_1(\mathbf{r}, t)$, taking (12) and (13) into account into the Routh functional and average the right sides of (6) over the "rapid" variables φ_2 and ϑ /8/. The results obtained will describe the evolution of the motion of a viscoelastic sphere in Andoyer variables.

Note that in the Routh functional (4) only two terms R_1 and Π_1, where

$$R_1 = {}^1/_2\,(\mathbf{G} - \mathbf{G}_u, J^{-1}[\mathbf{u}](\mathbf{G} - \mathbf{G}_u))$$

depend on the Andoyer canonical variables. We will represent these terms, apart from small terms of order ε and R^{-3}, in the form

$$\begin{aligned} R_1 &= R_0 - {}^1/_2\,\varepsilon A^{-2}(\mathbf{G}, J_1[\mathbf{u}_1]\mathbf{G}) - \varepsilon A^{-1}(\mathbf{G}_u', \mathbf{G}) \\ \Pi_1 &= D_1 - \mu\rho R^{-3}\varepsilon\int[3(O^{-1}\mathbf{R}^\circ, \mathbf{r})(O^{-1}\mathbf{R}^\circ, \mathbf{u}_1) - (\mathbf{r}, \mathbf{u}_1)]\,dx \\ D_1 &= -\mu M R^{-3} \\ \mathbf{G}_u' &= \int(\mathbf{r}\times\mathbf{u}_1^{\cdot})\rho dx, \quad (J_1[\mathbf{u}_1]\mathbf{G}, \mathbf{G}) \equiv 2\int[\mathbf{r}\times\mathbf{G}][\mathbf{u}_1\times\mathbf{G}]\rho dx \end{aligned} \tag{14}$$

In the system of coordinates $Cx_1x_2x_3$, integrally connected with the sphere, the vector $\mathbf{G}$ is constant in the unperturbed motion, and the function $\mathbf{u}_1(\mathbf{r}, t)$ depends on time through the variables $\varphi_2 = I_2A^{-1}t + \varphi_2(0)$, ϑ, and is 2π periodic in them.

Since the operation of averaging

$$\langle\cdot\rangle_{\varphi_2,\vartheta} = \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\cdot\, d\varphi_2\, d\vartheta$$

hence

$$\left\langle \mathbf{G}_u'\cdot\frac{\partial\mathbf{G}}{\partial p_k}\right\rangle_{\varphi_2,\vartheta} = \left\langle\int\mathbf{r}\times\left(\frac{\partial\mathbf{u}_1}{\partial\varphi_2}\varphi_2^{\cdot} + \frac{\partial\mathbf{u}_1}{\partial\vartheta}\vartheta^{\cdot}\right)\rho\, dx\,\frac{\partial\mathbf{G}}{\partial p_k}\right\rangle_{\varphi_2,\vartheta} = 0$$

where p_k takes values $I_1, I_2, I_3, \varphi_1, \varphi_2, \varphi_3$.

The term $-\varepsilon A^{-1}(\mathbf{G}_u', \mathbf{G})$ thus does not contribute in the averaging to the right sides of (6).

The second term in R_1 has the form

$$-{}^1/_2\,\varepsilon A^{-2}[(\mathbf{G}, J_1[\mathbf{u}_{11}]\mathbf{G}) + (\mathbf{G}, J_1[\mathbf{u}_{12}]\mathbf{G}) + (\mathbf{G}, J_1[\mathbf{u}_{13}]\mathbf{G})]$$

Since $\mathbf{u}_{12}(\mathbf{r})$ is a spherically symmetric function, we have

$$(\mathbf{G}, J_1[\mathbf{u}_{12}]\mathbf{G}) = {}^1/_2\, f_1(\mathbf{u}_{12})\, I_2^2$$

which results only in a perturbation of the rapid variable φ_2 in the respective equation of (6). The term

$$(\mathbf{G}, J_1[\mathbf{u}_{11}]\mathbf{G}) = (J_1[\rho I_2^2A^{-2}\mathbf{u}^*(\mathbf{r}^*)]\Gamma_1(\delta_2)\Gamma_3(\varphi_1)\mathbf{G}\quad \Gamma_1(\delta_2)\Gamma_3(\varphi_1)\mathbf{G})$$

where the second scalar product is given in a system of coordinates one of whose axis coincides with the vector $\mathbf{G}$. The function $\mathbf{u}^*(\mathbf{r}^*)$ is symmetric in this system of coordinates, and the operator J_1 has the form

$$J_1[\rho I_2^2A^{-2}\mathbf{u}^*(\mathbf{r}^*)] = \operatorname{diag}\{m_1, m_1, m_1\} + \operatorname{diag}\{0, 0, m_2 - m_1\}$$

where m_1, m_2 are constants.

Then

$$(G,\ J_1[u_{11}]G) = m_1 I_2^2 + (\mathrm{diag}\{0,0,m_2-m_1\}\Gamma_1(\delta_2)\Gamma_3(\varphi_1)G,$$
$$\Gamma_1(\delta_2)\Gamma_3(\varphi_1)G) = m_1 I_2^2 + (m_2-m_1)(G_1\sin\delta_2\sin\varphi_1 +$$
$$G_2\sin\delta_2\cos\varphi_1 + G_3\cos\delta_2)^2$$
$$G_1 = \sqrt{I_2^2-I_1^2}\sin\varphi_1,\quad G_2 = \sqrt{I_2^2-I_1^2}\cos\varphi_1,\quad G_3 = I_1$$

from which it follows that

$$\frac{\partial}{\partial\varphi_1}(G, J_1[u_1]G) = (m_2-m_1)\,2\,(G_1\sin\delta_2\sin\varphi_1 +$$
$$G_2\sin\delta_2\cos\varphi_1 + G_3\cos\delta_2)\,(\sqrt{I_2^2-I_1^2}\cos\varphi_1\sin\delta_2\sin\varphi_1 -$$
$$\sqrt{I_2^2-I_1^2}\sin\varphi_1\sin\delta_2\cos\varphi_1) = 0$$
$$\frac{\partial}{\partial\varphi_2}(G, J_1[u_1]G) = \frac{\partial}{\partial\varphi_3}(G, J_1[u_1]G) = 0$$
$$\frac{\partial}{\partial I_1}(G, J_1[u_1]G) = 2(m_2-m_1)(G_1\sin\delta_2\sin\delta_1 +$$
$$G_2\sin\delta_2\cos\varphi_1 + G_3\cos\delta_2)\left(\frac{-I_1}{\sqrt{I_2^2-I_1^2}}\sin\varphi_1\sin\delta_2\sin\varphi_1 -\right.$$
$$\left.\frac{I_1}{\sqrt{I_2^2-I_1^2}}\cos\varphi_1\sin\delta_2\cos\varphi_1 + \cos\delta_2\right) = 0$$
$$\frac{\partial}{\partial I_2}(G, J_1[u_1]G) = 2m_1I_2 + 2(m_2-m_1)I_2 = 2m_2I_2$$
$$\frac{\partial}{\partial I_3}(G, J_1[u_1]G) = 0$$

Thus the non-zero term provides a correction only to the derivative of the rapid variable φ_2.

Further

$$(G, J_1[u_{13}]G) = \left(G, \left(J_1[u_{130}] - \chi J_1\left[\frac{\partial u_{130}}{\partial\vartheta}\right]\vartheta^\cdot - \chi J_1\left[\frac{\partial u_{130}}{\partial\varphi_2}\right]\varphi_2^\cdot\right)G\right)$$

When averaging over ϑ and φ_2 the terms containing $\vartheta^\cdot$ and $\varphi_2^\cdot$ provide only zero, since the function $u_{130}(r, \vartheta, \varphi_2)$ is 2π periodic in ϑ and φ_2. It remains to consider the term

$$(G,\ J_1[u_{130}]G) = (J_1[-3\mu\rho R^{-3}u^*(r^*)]\,O_1(t)\,G,\ O_1(t)\,G)$$

where the second scalar product is written in the orbital system of coordinates.

Since

$$J_1[-3\mu\rho R^{-3}u^*(r^*)] = \mathrm{diag}\{l_1, l_1, l_1\} + \mathrm{diag}\{0, 0, l_2 - l_1\}$$

we have

$$(G, J_1[u_{130}]G) = l_1 I_2^2 + (\mathrm{diag}\{0,0,l_2-l_1\}\,O_1(t)\,G, O_1(t)G) =$$
$$l_1 I_2^2 + (l_2-l_1)(G_1\gamma_{31} + G_2\gamma_{32} + G_3\gamma_{33})^2$$

where $O_1(t) = (\gamma_{ij})$, l_1, l_2 are constants, and $(\gamma_{31}, \gamma_{32}, \gamma_{33})$ are components of the unit vector e_z on the axis of the orbital system of coordinates Cz in the system of coordinates $Cx_1x_2x_3$, attached to the sphere. We shall show that in calculating partial derivatives of the expression $(G, J_1[u_{130}]G)$, only the derivative with respect to I_2 is different from zero. The vector G is independent of $I_3, \varphi_2, \varphi_3$, and its derivatives $\partial G/\partial\varphi_1$ and $\partial G/\partial I_1$ are orthogonal to G. We have

$$\frac{\partial}{\partial\varphi_1}(G, J_1[u_{130}]G) = 2(l_2-l_1)(G, e_z)\left(\frac{\partial G}{\partial\varphi_1}, e_z\right)$$
$$\frac{\partial}{\partial I_1}(G, J_1[u_{130}]G) = 2(l_2-l_1)(G, e_z)\left(\frac{\partial G}{\partial I_1}, e_z\right)$$

The pair of vectors $G, \partial G/\partial\varphi_1$ and $G, \partial G/\partial I_1$ is orthogonal to and fixed in the system of coordinates $Cx_1x_2x_3$, and the vector e_z rotates at constant angular velocity connected with the variation of the angles φ_2 and ϑ. It follows from this that the projection of the vector e_z rotates at constant velocity in the plane which contains vectors $G, \partial G/\partial\varphi_1$ and $G, \partial G/\partial I_1$, and its product with mutually orthogonal vectors, yields zero as a result of the averaging operation.

Thus the term R_1 in the Routh functional yields, after averaging in the equations of perturbed motion (6), terms that are non-zero only in the equation for the rapid angular variable φ_2. Consequently, the evolution of slow variables I_i $(i = 1, 2, 3)$, φ_1, φ_3 will be determined by the term Π_1 in the Routh functional (4).

By relation (14), on the right sides of (6) there will remain from Π_1 only the derivatives of the quantities

$$-3\mu\rho R^{-3}\varepsilon\sum_{k=1}^{3}\int(R^\circ, Or)(R^\circ, Ou_{1k})\,dx$$

where the operator

$$O = \Gamma_3(\varphi_3)\,\Gamma_1(\delta_1)\;\Gamma_3(\varphi_2)\;\Gamma_1(\delta_2)\,\Gamma_3(\varphi_1)$$

depends on the Andoyer variables.

Since $\mathbf{u}_{12}(\mathbf{r})$ is a spherically symmetric function, the term

$$\int (O^{-1}\mathbf{R}^\circ,\ \mathbf{r})\ (O^{-1}\mathbf{R}^\circ,\ \mathbf{u}_{12}(\mathbf{r}))\,dx$$

is independent of the vector $O^{-1}\mathbf{R}^\circ$ and, consequently, does not affect the right sides of (6).

Consider the term

$$P_1 = \int (O^{-1}\mathbf{R}^\circ, \mathbf{r})\ (O^{-1}\mathbf{R}^\circ,\ \mathbf{u}_{11}(\mathbf{r}))\,dx = \int (\Gamma_1(\delta_2)\Gamma_3(\varphi_1)\,O^{-1}\mathbf{R}^\circ,\ \mathbf{r}^*)\ (\Gamma_1(\delta_2)\,\Gamma_3(\varphi_1)\,O^{-1}\mathbf{R}^\circ,\ \mathbf{u}^*(\mathbf{r}^*))\,\rho I_2^2A^{-2}dx$$

Denoting the coordinates of the vector $\Gamma_1(\delta_2)\Gamma_3(\varphi_1)O^{-1}\mathbf{R}^0$ by $(\alpha_1, \alpha_2, \alpha_3)$, and using relation (12), we obtain

$$P_1 = \rho I_2^2A^{-2}\,[(\alpha_1^2+\alpha_2^2)\,(b_1f_1+b_1f_2+b_2f_2+c_1f_3) + \alpha_3^2\,(2a_1f_2+a_2f_1+c_2f_3)]$$

$$f_1 = \int x^4\,dx,\ f_2 = \int x^2y^2dx,\ f_3 = \int x^2dx$$

Since $\alpha_1^2+\alpha_2^2+\alpha_3^2 = 1$, we have

$$P_1 = D_2' + D_2\alpha_3^2 = D_2' + D_2\,(\Gamma_1(\delta_2)\Gamma_3(\varphi_1)\,[\Gamma_3(-\varphi_1)\times \Gamma_1(-\delta_2)\,\Gamma_3(-\varphi_2)\,\Gamma_1(-\delta_1)\;\Gamma_3(-\varphi_3)]\,\Gamma_0^{-1}(\vartheta)\,\mathbf{e}_z,\ \mathbf{e}_z)^2 \tag{15}$$

$$D_2 = \rho I_2^2A^{-2}\,[(a_2-b_1)\,f_1 + (2a_1-b_1-b_2)\,f_2 + (c_2-c_1)\,f_3]$$

$$D_2' = \rho I_2^2A^{-2}\,(b_1f_1+b_1f_2+b_2f_2+c_1f_3),\ \ \mathbf{e}_z = (0,0,1)$$

Note that the Andoyer variables by which we calculate the partial deriatives in (6), are contained only in the terms in the brackets, i.e.

$$\frac{\partial P_1}{\partial\cdot} = 2D_2\,(\Gamma_1(-\delta_1)\,\Gamma_3(-\varphi_3)\,\Gamma_0^{-1}(\vartheta)\,\mathbf{e}_z,\mathbf{e}_z)\,(\Gamma_1(\delta_2)\,\Gamma_3(\varphi_1)\ \times \frac{\partial}{\partial\cdot}\,[\Gamma_3(-\varphi_1)\,\Gamma_1(-\delta_2)\,\Gamma_3(-\varphi_2)\,\Gamma_1(-\delta_1)\,\Gamma_3(-\varphi_3)]\times\Gamma_0^{-1}(\vartheta)\,\mathbf{e}_z,\mathbf{e}_z)$$

where ∂/∂ denotes the partial derivative with respect to one of variables $\varphi_1, \varphi_2, \varphi_3, I_1, I_3$. Calculating the partial derivatives and averaging over φ_2 and ϑ, we obtain

$$\left\langle\frac{\partial P_1}{\partial\varphi_1}\right\rangle_{\varphi_2,\vartheta} = \left\langle\frac{\partial P_1}{\partial\varphi_2}\right\rangle_{\varphi_2,\vartheta} = \left\langle\frac{\partial P_1}{\partial\varphi_3}\right\rangle_{\varphi_2,\vartheta} = \left\langle\frac{\partial P_1}{\partial I_1}\right\rangle_{\varphi_2,\vartheta} = 0 \tag{16}$$

$$\left\langle\frac{\partial P_1}{\partial I_3}\right\rangle_{\varphi_2,\vartheta} = -D_2I_2^{-1}\cos\delta_1$$

Let us calculate the partial derivatives with respect to Andoyer variables of the expression

$$P_2 = \int (O^{-1}\mathbf{R}^\circ, \mathbf{r})\ (O^{-1}\mathbf{R}^\circ,\ \mathbf{u}_{13}(\mathbf{r}, t))\,dx \tag{17}$$

The integral in (17) is conveniently calculated in the orbital system of coordinates since the function $\mathbf{u}_{13}(\mathbf{r}, t)$ has the simplest form in that system. The operator of transferring from axes attached to the sphere to orbital axes is

$$O_1(t) = L = \Gamma_0(\vartheta)\,O,\ \ O = \Gamma_3(\varphi_3)\Gamma_1(\delta_1)\;\Gamma_3(\varphi_2)\;\Gamma_1(\delta_2)\,\Gamma_3(\varphi_1)$$

and then

$$P_2 = \int (LO^{-1}\mathbf{R}^\circ, \mathbf{r}')\ (LO^{-1}\mathbf{R}^\circ,\ \mathbf{u}_{13}'(\mathbf{r}', t))\,dx$$

where $\mathbf{u}_{13}'(\mathbf{r}', t)$ is represented by formula (13), and

$$LO^{-1}\mathbf{R}^c = \mathbf{e}_z\,(\gamma_1, \gamma_2, \gamma_3);\ \gamma_1 = \gamma_2 = 0,\ \gamma_3 = 1$$

However later on it is necessary to calculate derivatives with respect to Andoyer variables of the matrix O^{-1}, and hence in evaluating the integral we retain the notation for the components of vector $\mathbf{e}_z\,(\gamma_1, \gamma_2, \gamma_3)$. First of all we obtain from (13)

$$S = \begin{Vmatrix} 0 & g_1 & g_2 \\ -g_1 & 0 & g_3 \\ -g_2 & -g_3 & 0 \end{Vmatrix} \tag{18}$$

$$g_1 = \sin\delta_1\sin(\vartheta-\varphi_3)\,\dot\varphi_2,\ \ g_2 = \cos\delta_1\dot\varphi_2 - \dot\vartheta,\ \ g_3 = \sin\delta_1\cos(\vartheta-\varphi_3)\,\dot\varphi_2$$

and then

$$\mathbf{u}_2'(\mathbf{r}', t) = \alpha \mathbf{u}^*(\mathbf{r}^*) - \alpha\chi [(b_1 - a_1)(x^2 + y^2) + (b_2 - a_2)z^2 + c_1 - c_2](g_2 z,\ g_3 z,\ g_2 x + g_3 y) - 2\alpha\chi [(b_1 - b_2)(x,\ y,\ 0) + (a_1 - a_2)(0,\ 0,\ z)](g_2 xz + g_3 yz),\quad \alpha = -3\mu\rho R^{-3}$$

As a result, we obtain

$$\begin{aligned} P_2 &= (\gamma_1^2 + \gamma_2^2) D_3 + \gamma_3^2 D_4 - \gamma_1\gamma_3 D_5 g_2 - \gamma_2\gamma_3 D_5 g_3 \\ D_3 &= \alpha \int [b_1 (x^2 + y^2) + b_2 z^2 + c_1] x^2 dx \\ D_4 &= \alpha \int [a_1 (x^2 + y^2) + a_2 z^2 + c_2] z^2 dx \\ D_5 &= \alpha\chi \int \{[(b_1 - a_1)(x^2 + y^2) + (b_2 - a_2) z^2 + c_1 - c_2] \times (x^2 + y^2) + 2(a_1 - a_2 + b_1 - b_2) x^2 z^2\} dx \end{aligned} \tag{19}$$

Then

$$\begin{aligned} &\frac{\partial \mathbf{e}_z}{\partial \varphi_1} = L \frac{\partial O^{-1}}{\partial \varphi_1} \mathbf{R}^\circ \Rightarrow \left\langle \frac{\partial \mathbf{e}_z}{\partial \varphi_1} \right\rangle_{\varphi_2} = (-\cos\delta_2 \cos\delta_1, -\sin\delta_1 \cos\delta_2 \cos(\vartheta - \varphi_3), 0) \\ &\frac{\partial \mathbf{e}_z}{\partial \varphi_2} = L \frac{\partial O^{-1}}{\partial \varphi_2} \mathbf{R}^\circ \Rightarrow \left\langle \frac{\partial \mathbf{e}_z}{\partial \varphi_2} \right\rangle_{\varphi_2} = (-\cos\delta_1, -\cos(\vartheta - \varphi_3)\sin\delta_1, 0) \\ &\frac{\partial \mathbf{e}_z}{\partial \varphi_z} = L \frac{\partial O^{-1}}{\partial \varphi_3} \mathbf{R}^\circ \Rightarrow \left\langle \frac{\partial \mathbf{e}_z}{\partial \varphi_3} \right\rangle_{\varphi_2} = (1, 0, 0) \\ &\frac{\partial \mathbf{e}_z}{\partial I_1} = L \frac{\partial O^{-1}}{\partial \delta_2} \mathbf{R}^\circ \frac{\partial \delta_2}{\partial I_1} \Rightarrow \left\langle \frac{\partial \mathbf{e}_z}{\partial I_1} \right\rangle_{\varphi_2} = (0, 0, 0) \\ &\frac{\partial \mathbf{e}_z}{\partial I_3} = L \frac{\partial O^{-1}}{\partial \delta_1} \mathbf{R}^\circ \frac{\partial \delta_1}{\partial I_3} \Rightarrow \left\langle \frac{\partial \mathbf{e}_z}{\partial I_3} \right\rangle_{\varphi_2} = (0, 0, 0) \end{aligned} \tag{20}$$

In calculating the expressions in (20) the averaging was carried out over the angle φ_2, since this angle does not appear in g_2 and g_3 in (18). Calculating the partial derivatives with respect to canonical variables of P_2, averaging the results over φ_2 and ϑ and taking into account the relations (18)–(20), we obtain

$$\begin{aligned} &\left\langle \frac{\partial P_2}{\partial \varphi_1} \right\rangle_{\varphi_2,\vartheta} = D_5 \frac{I_1}{A I_2} \left(\frac{I_2^2 + I_3^2}{2I_2} - \frac{I_3}{I_2} A\vartheta^\cdot \right) \\ &\left\langle \frac{\partial P_2}{\partial \varphi_2} \right\rangle_{\varphi_2,\vartheta} = \frac{D_5}{A} \left(\frac{I_2^2 + I_3^2}{2I_2} - \frac{I_3}{I_2} A\vartheta^\cdot \right) \\ &\left\langle \frac{\partial P_2}{\partial \varphi_3} \right\rangle_{\varphi_2,\vartheta} = -\frac{D_5}{A} (I_3 - A\vartheta^\cdot); \quad \left\langle \frac{\partial P_2}{\partial I_1} \right\rangle_{\varphi_2,\vartheta} = \left\langle \frac{\partial P_2}{\partial I_3} \right\rangle_{\varphi_2,\vartheta} = 0 \end{aligned} \tag{21}$$

As the result of averaging, we obtain from (6) approximate equations that define the evolution of the viscoelastic sphere rotational motion

$$\begin{aligned} &I_1^\cdot = -k_1 \frac{I_1}{I_2} \left(\frac{I_2^2 + I_3^2}{2I_2} - \frac{I_3}{I_2} A\vartheta^\cdot \right) \\ &I_2^\cdot = -k_1 \left(\frac{I_2^2 + I_3^2}{2I_2} - \frac{I_3}{I_2} A\vartheta^\cdot \right) \\ &I_3^\cdot = -k_1 (I_3 - A\vartheta^\cdot) \\ &\varphi_1^\cdot = 0, \quad \varphi_3^\cdot = -k_2 I_3 \\ &k_1 = \frac{9\mu^2\rho^2\varepsilon\chi}{R^8 A} \int \{[(b_1 - a_1)(x^2 + y^2) + (b_2 - a_2) z^2 + c_1 - c_2](x^2 + z^2) + 2(a_1 - a_2 + b_1 - b_2) x^2 z^2\} dx > 0 \\ &k_2 = \frac{R^3}{6\mu A\chi} k_1 > 0 \end{aligned} \tag{22}$$

From (22) it follows that I_3 approaches $A\vartheta^\cdot$, and

$$I_3 = A\vartheta^\cdot + (I_3(0) - A\vartheta^\cdot) e^{-k_1 t} \tag{23}$$

Then

$$I_2^2 = A^2\vartheta^{\cdot 2} + (I_2^2(0) - I_3^2(0) - 2I_3(0) A\vartheta^\cdot - 2A^2\vartheta^{\cdot 2}) e^{-k_1 t} + (I_3(0) - A\vartheta^\cdot)^2 e^{-2k_1 t} \tag{24}$$

Relation (24) determines the law of variation of I_2 which also approaches $A\vartheta^\cdot$. The angle δ_2 between the axis Cx_3 and the vector $\mathbf{G}$ does not vary, since by (22)

$$(\cos\delta_2)^\cdot = -\sin\delta_2 \frac{I_1^\cdot I_2 - I_2^\cdot I_1}{I_2^2} = 0$$

This means that I_1 varies in proportion to the variation of I_2

$$I_1(t) = I_2(t) I_1(0) / I_2(0)$$

Since the angles φ_1 and δ_2 are constant, the axis of rotation of the sphere is fixed in the system of coordinates which is integrally connected to the sphere.

In the course of evolution the axis of rotation of a viscoelastic sphere approaches the normal to the orbital plane and the angular velocity of the sphere approaches the orbital angular velocity.

Two small parameter k_1 and k_2 occur in (22), and $k_1 \ll k_2$. Consequently, in the rotational motion of the sphere there are two evolutions, a rapid and a slow one. The rapid and a slow one. The rapid evolution is defined by (22) with $k_1 = 0$, when only the angle φ_3 varies and the vector G describes a circular cone with axis of symmetry that coincides with the normal to the orbit. The slow evolution corresponds to terms that contain k_1 in (22) and defines the variation of the quantities I_1, I_2, I_3.

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THE PROBLEM OF THE OPTIMUM RAPID BRAKING OF AN AXISYMMETRIC SOLID ROTATING AROUND ITS CENTRE OF MASS*

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The problem of the braking of a solid with an axisymmetric ellipsoid of inertia using three pairs of jet engines producing control moments directed along the principal axes of the ellipsoid of inertia /1–4/ is considered. The structure of the optimal trajectories is analyzed. It is shown that the four rays that lie in the plane normal to the axis of dynamic symmetry are not only the phase trajectories with special control /3/, but perform the part of main lines. The optimal trajectories reach the main lines after an infinite number of control reversals. Such trajectories which reach the main lines fill, in phase space, the outer region of two intersecting circular cones encircling the axis of dynamic symmetry.

1. Statement of the problem and formulation of the basis results.

The problem of the most rapid braking of the rotation of a solid with an axisymmetric ellipsoid of inertia can be formulated as follows /1/. The system of Euler equations is given in the normal form

$$x^\cdot = b_1u_1,\quad y^\cdot = -Dxz + b_2u_2,\quad z^\cdot = Dxy + b_3u_3;\quad D = (A - C)/B,\ B = A \tag{1.1}$$

with constraints

$$|u_i| \leqslant 1,\ i = 1, 2, 3 \tag{1.2}$$

where x, y, z are the projections of the vector of the instantaneous angular velocity of the solid in a moving system of coordinates attached to the principal axes of the central ellipsoid of inertia, u_i are the controls, b_i are constants, and A, B, C are the moments of inertia. It is required to transfer an arbitrary phase point of system (1.1), whose coordinates at the instant $t = 0$ are denoted by (x_0, y_0, z_0), into the origin of coordinates in the minimum time.

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